Problem proposed by Arkady Alt, San Jose, California, USA.

Let $\Delta(x,y,z) := 2(xy+yz+xz)-(x^2+y^2+z^2)$. and a,b,c be sidelengths of a triangle ABC. Prove that

$$F^2 \geq \frac{3}{16} \cdot \frac{\Delta(a^3,b^3,c^3)}{\Delta(a,b,c)},$$

where F is area of $\triangle ABC$.

Solution.

Due to homogeneity of inequality assume that semiperimeter s equal 1. Then denoting $x := 1-a, y := 1-b, z := 1-c$ and $p = xy+yz+zx, q := xyz$ we obtain $a = 1-x, b = 1-y, c = 1-z$ where $x+y+z = 1, x,y,z > 0$.

Then $\Delta(a,b,c) = 4p,\ F^2 = 16q,\ \Delta(a^3,b^3,c^3) = 36q-18pq-9p^2+4p^3+3q^2$ and, therefore, $h(p,q) := \Delta(a,b,c) \cdot \Delta(a^2,b^2,c^2) - 3\,\Delta(a^3,b^3,c^3) =$
$4p \cdot 16q - 3(36q-18pq-9p^2+4p^3+3q^2) = p^2(27-12p)-(108-118p)q-9q^2$.

Since system of equations

$$\begin{cases} a+b+c = 1 \\ ab+bc+ca = p \\ abc = q \end{cases}$$

have solutions in nonnegative a,b,c iff $p,q \geq 0$ and

(**1**) $\quad 27q^2-2(9p-2)q+4p^3-p^2 \leq 0$.

Since (1) yields $1-3p \geq 0$ then denoting $t := \sqrt{1-3p}$ we obtain $p = \dfrac{1-t^2}{3},\ 0 \leq t < 1$ and for (**1**) effective equivalent form

(**2**) $\quad \max\left\{0, \dfrac{(1+t)^2(1-2t)}{27}\right\} \leq q \leq \dfrac{(1-t)^2(1+2t)}{27}$.

Since $108-118p > 0$ for $p \leq \frac{1}{3} \Leftrightarrow 0 \leq t < 1$ and $q \leq \dfrac{(1-t)^2(1+2t)}{27}$ then

$$h(p,q) = h\left(\frac{1-t^2}{3}, q\right) \geq h\left(\frac{1-t^2}{3}, \frac{(1-t)^2(1+2t)}{27}\right) =$$

$$\frac{(4t^2+23)(1-t^2)^2}{9} - \left(108 - \frac{118(1-t^2)}{3}\right)\frac{(1-t)^2(1+2t)}{27} - 9\left(\frac{(1-t)^2(1+2t)}{27}\right)^2 =$$

$$\frac{32}{81}t^2(4-t)(1-t)^3 \geq 0.$$